(15) Show that Theorem 2.36 and its Corollary become false (in $\mathbb{R}^1$, for examp
if the word "compact" is replaced by "closed" or by "bounded".

Solution: We want to exhibit a collection $\{A_\alpha\}$ of closed subsets of $\mathbb{R}$ such that the intersection of every finite subcollection of $\{A_\alpha\}$ is non empty but $\cap A_\alpha = \emptyset$. Then we want to do the same but for $\{B_\alpha\}$ a collection of bounded subsets of $\mathbb{R}$.

(i) For closed: Consider $A_n = \{n, n+1, n+2, \ldots\}$. The collection $\{A_n\}_{n\in\mathbb{N}}$ has the finite intersection property since if you pick an arbitrary subcollection of $\{A_n\}$, say $\{A_k\}_{k\in I}$; s.t. $I \subset \mathbb{N}$ and $|I|<\infty$, then $M=\max(I)$, which we know exis since I is finite, is such that $M \in \bigcap_{k\in I} A_k$. Moreover, A_n is closed for $n\in\mathbb{N}$ since each A_n contains all of its limit points (there are no limit points in A_n so these are contained in A_n trivially). However, $\bigcap_{n\in\mathbb{N}} A_n = \emptyset$. Suppose $m \in \bigcap_{n\in\mathbb{N}} A_n$ then $m \geq n$; for all $n \in \mathbb{N}$. So m is a bound for $\mathbb{N}$, which we know does not exists. Therefore, there is no such m.
Note that $A_{n+1} \subset A_n$; and $A_n \neq \emptyset$ for all n. So this same collection of closed sets serve as a counterexample of Corollary to theorem 2.36 if we replace the word "compact" by "closed".

(ii) For bounded: Consider $A_n = (0, \frac{1}{n})$. The collection $\{A_n\}_{n\in\mathbb{N}}$ has the fini intersection property since if you pick an arbitrary subcollection of $\{A_n\}$, sa $\{A_k\}_{k\in I}$, where $I \subset \mathbb{N}$ and $|I|<\infty$, then $M=\max(I)$, which we know exists because I is finite, is such that $\frac{1}{M+1} \in \bigcap_{k\in I} A_k$. Precisely, $0 < \frac{1}{M+1} < \frac{1}{M} \leq \frac{1}{k}$ for $k \in I$, hence $0 < \frac{1}{M+1} < \frac{1}{k} \Rightarrow \frac{1}{M+1} \in A_k$ for all $k \in I$. Moreover, A_n bounded for $n\in\mathbb{N}$. Just pick $R=2$ and $x=1$ and then $A_n \subset N_R(x)$. However, $\bigcap_{n\in\mathbb{N}} A_n = \emptyset$. Suppose there exists $x \in \bigcap_{n\in\mathbb{N}} A_n$. Then $0 < x < \frac{1}{n}$, for all $n \in$
Since $x>0$ and $\frac{1}{n}>0$ we can apply the archimedian property to conc that there exists $m\in\mathbb{N}$, such that $m \cdot x > \frac{1}{n} \Rightarrow x > \frac{1}{m \cdot n}$ and thus $x \notin A_{m\cdot n}$ therefore $x \notin \bigcap_{n\in\mathbb{N}} A_{m\cdot n}$, a contradiction. Hence, there is no such x
So $\bigcap_{n\in\mathbb{N}} A_n = \emptyset$. Note that $A_{n+1} = (0, \frac{1}{n+1}) \subset (0, \frac{1}{n}) = A_n \Rightarrow A_{n+1} \subset A_n$ for all $n \in \mathbb{N}$
So this counters the corollary if we replace "compact" by bounded.

9). (a). If A and B are disjoint closed sets in some metric space X, rove that they are separated.

f: Let $A \subseteq (X,d)$, $B \subseteq (X,d)$, A, B closed sets such that $A \cap B = \emptyset$.
e want to prove (i) $A \cap \overline{B} = \emptyset$ and (ii) $\overline{A} \cap B = \emptyset$.

) Suppose $A \cap \overline{B} \neq \emptyset$. Let $x \in A \cap \overline{B}$. By definition, $x \in A$ and $x \in \overline{B}$.
ut $x \in \overline{B} \Leftrightarrow x \in B \cup B'$.
If $x \in B$ <u>then</u> $x \in A$ and $x \in B \Leftrightarrow x \in A \cap B$; but $A \cap B = \emptyset$, a contradiction.
) If $x \in B'$ <u>then</u> $x \in B$ since B is closed. But then $x \in A \cap B$; a contradiction.
any case we get a contradiction and thus, $\boxed{A \cap \overline{B} = \emptyset}$

) Suppose $\overline{A} \cap B \neq \emptyset$. Let $y \in \overline{A} \cap B$. By definition, $y \in \overline{A}$ and $y \in B$.
t $y \in \overline{A} \Leftrightarrow y \in A \cup A'$.
If $y \in A$ <u>then</u> $y \in A$ and $y \in B \Leftrightarrow y \in A \cap B$; but $A \cap B = \emptyset$, a contradiction.
If $y \in A'$ <u>then</u> $y \in A$ since A is closed. But then $x \in A \cap B$; a contradiction.
any case we get a contradiction and thus, $\boxed{\overline{A} \cap B = \emptyset}$
& (ii) $\Rightarrow$ $A \cap \overline{B} = \overline{A} \cap B = \emptyset \Leftrightarrow$ A and B are separated.

) (b) Prove the same for disjoint open sets.
: Let $A \subseteq (X,d)$, $B \subseteq (X,d)$, A, B open sets such that $A \cap B = \emptyset$.
want to prove (i) $A \cap \overline{B} = \emptyset$ and (ii) $\overline{A} \cap B = \emptyset$
Suppose $A \cap \overline{B} \neq \emptyset$. Let $x \in A \cap \overline{B}$. By definition, $x \in A$ and $x \in \overline{B}$. $\Leftrightarrow x \in B \cup B'$.
$x \in B$ <u>then</u> $x \in A$ and $x \in B \Leftrightarrow x \in A \cap B$; But $A \cap B = \emptyset$, a contradiction.
if $x \in B'$ <u>then</u> x is a limit point of B. But $x \in A \Rightarrow x$ is interior to A.
e A is open. Hence, $\exists\, r > 0$ s.t. $N_r(x) \subset A$. But for that same r we
e that $N_r(x) \setminus \{x\} \cap B \neq \emptyset$ (since x is a limit point of B). Consider y s.t.
$N_r(x) \setminus \{x\} \cap B$. then $y \in N_r(x)$, $y \neq x$, $y \in B$. Moreover $y \in N_r(x) \subset A$ and thus
A, which means that $y \in A$ and $y \in B \Leftrightarrow y \in A \cap B$; a contradiction.
any case we get a contradiction and thus $\boxed{A \cap \overline{B} = \emptyset}$
Suppose $\overline{A} \cap B \neq \emptyset$. Let $x \in \overline{A} \cap B$. By definition, $x \in \overline{A}$ and $x \in B$. Hence, $x \in A \cup A'$.
$x \in A$ <u>then</u> $x \in A$ and $x \in B \Leftrightarrow x \in A \cap B$; but $A \cap B = \emptyset$, a contradiction.
f $x \in A'$ <u>then</u> x is a limit point of A. But $x \in B \Rightarrow x$ is interior to B. Since
s open. Hence, $\exists\, r > 0$ s.t. $N_r(x) \subset B$. But for that same r we have that
$x) \setminus \{x\} \cap A \neq \emptyset$. (since x is a limit point of A). Consider y s.t.

$\longrightarrow$

$y \in N_r(x)\setminus\{x\} \cap A$. then $y \in N_r(x)$, $y \neq x$, $y \in A$. Moreover $y \in N_r(x) \subset B$ and thus $y \in B$, which means that $y \in A$ and $y \in B \Leftrightarrow y \in A \cap B$; a contradiction.

In any case we get a contradiction and thus $\boxed{\bar{A} \cap B = \emptyset}$

(i) & (ii) $\Rightarrow A \cap \bar{B} = \bar{A} \cap B = \emptyset \Leftrightarrow$ A and B are separated.

(c) Fix $p \in X$, $\delta > 0$. Define $A = \{q \in X : d(p,q) < \delta\}$ and $B = \{q \in X : d(p,q) > \delta$

Prove that A and B are separated.

Pf: By theorem (2.19) A is an open set since A is a neighborhood. If we can prove that B is open and $A \cap B = \emptyset$, then by (b) we are done.

(i) $A \cap B = \emptyset$, since if $x \in A \cap B$ then $d(x,p) < \delta$ and $d(x,p) > \delta$, a contradic

So there is no such x, which means that A and B are disjoint.

(ii) B is open. We will prove that If $x \in B$ then x is interior to B.

Let $x \in B$. Pick r s.t. $r = d(p,x) - \delta$. Since $x \in B$, $d(p,x) > \delta$ so $r > 0$. We want to prove that $N_r(x) \subset B$. So let $y \in N_r($

Look at $N_r(x)$. We want to prove that $N_r(x) \subset B$.

then $d(y,x) < r = d(p,x) - \delta \Rightarrow d(y,x) < d(p,x) - \delta$

$\Rightarrow d(y,x) - d(p,x) < -\delta \Rightarrow d(p,x) - d(y,x) > \delta$. But by triangula

inequality: $d(p,x) \leq d(p,y) + d(y,x) \Rightarrow d(p,x) - d(y,x) \leq d(p,y)$. therefore

$d(p,y) \geq d(p,x) - d(y,x) > \delta \Rightarrow d(p,y) > \delta \Rightarrow y \in B \Rightarrow N_r(x)$

So for every $x \in B$ there exists $r > 0$ s.t. $N_r(x) \subset B$. So x is interior to B.

Since x was arbitrary in B, we can conclude that B is open.

By (i), (ii), and part (b) we conclude that A and B are separated.

(d) Prove that every connected metric space with at least two points is uncountable

Pf: Let X be a connected metric space such that $|X| \geq 2$. Let $p, q \in X$ b different points in X. $p \neq q$. then, $d(p,q) > 0$. Define $\delta_\gamma = \gamma d(p,q)$ where $\gamma \in (0,1)$, so that $\delta_\gamma > 0$. Consider:

$A_\gamma = \{x \in X : d(x,p) < \delta_\gamma\}$ and $B_\gamma = \{x \in X : d(x,p) > \delta_\gamma\}$.

Note that $d(p,p) = 0 < \delta_\gamma \Rightarrow p \in A_\gamma$ and $d(p,q) > \delta_\gamma = \gamma d(p,q)$ (Recall $\gamma \in (0,1$ $\Rightarrow q \in B_\gamma$. Hence, $A_\gamma \neq \emptyset$ and $B_\gamma \neq \emptyset$ for any γ. By part (c) we know t A_γ and B_γ are separated, for each γ. Moreover, since X is a connected me space; $X \neq A_\gamma \cup B_\gamma$. So for each $\gamma \in (0,1)$, there exists a point $r_\gamma \in X$ s that $r_\gamma \notin A$ and $r_\gamma \notin B_\gamma$. So $d(r_\gamma,p) \geq \delta_\gamma$ and $d(r_\gamma,p) \leq \delta_\gamma \Rightarrow d(r_\gamma,p) = \delta$

o, for each $\gamma \in (0,1)$ we have found an element $r_\gamma \in X$. Note that for two istinct values of γ we get at least two distinct points r_γ in X, since if $\gamma_1, \gamma_2 \in (0,1)$ re such that $\gamma_1 \neq \gamma_2 \Rightarrow d(p, r_{\gamma_1}) = \gamma_1 \neq \gamma_2 = d(p, r_{\gamma_2})$, by triangular inequality nd the fact that the reals are ordered so we can assume without loss generality that $\gamma_1 > \gamma_2$:

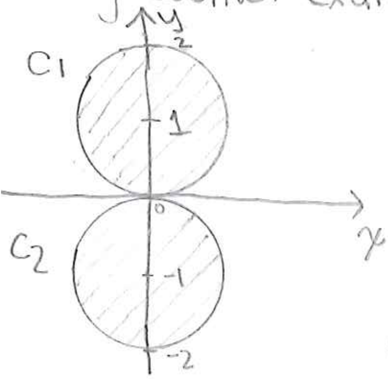
$$d(p, r_{\gamma_1}) \leq d(p, r_{\gamma_2}) + d(r_{\gamma_2}, r_{\gamma_1}) \Rightarrow d(p, r_{\gamma_1}) - d(p, r_{\gamma_2}) \leq d(r_{\gamma_2}, r_{\gamma_1})$$

$$\Rightarrow \gamma_1 - \gamma_2 \leq d(r_{\gamma_2}, r_{\gamma_1}) \Rightarrow d(r_{\gamma_2}, r_{\gamma_1}) > 0 \Rightarrow r_{\gamma_2} \neq r_{\gamma_1}$$

w, build the function $f: (0,1) \to X$, as follow $f(\gamma) = r_\gamma$. this is an injection, $\gamma_1, \gamma_2 \in (0,1)$ are such that $f(\gamma_1) = f(\gamma_2)$ then $r_{\gamma_1} = r_{\gamma_2} \Rightarrow$ $d(p, r_{\gamma_1}) = \gamma_1 = d(p, r_{\gamma_2}) = \gamma_2 \Rightarrow \gamma_1 = \gamma_2$. therefore, the set X contains an ncountable set $\{r_\gamma\} \subset X$; $\gamma \in (0,1)$. So X has to be uncountable.

) Are closures and interiors of connected sets always connected?

: ⓐ interiors of connected sets may not be connected. Consider the llowing counter example in $\mathbb{R}^2$; withe the usual metric:



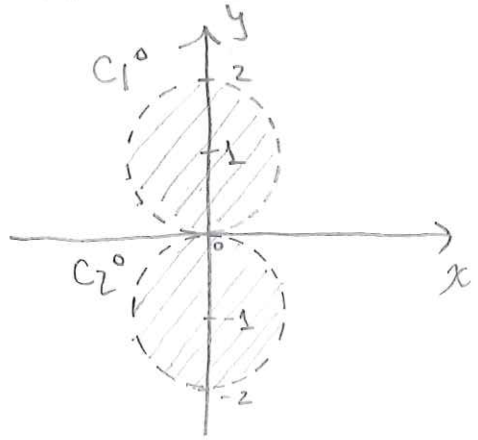
$$C_1 = \{(x,y) \in \mathbb{R}^2 \mid x^2 + (y-1)^2 \leq 1\}$$

$$C_2 = \{(x,y) \in \mathbb{R}^2 \mid x^2 + (y+1)^2 \leq 1\}$$

the interiors of these sets are:

$$C_1^\circ = \{(x,y) \in \mathbb{R}^2 \mid x^2 + (y-1)^2 < 1\}$$

$$C_2^\circ = \{(x,y) \in \mathbb{R}^2 \mid x^2 + (y+1)^2 < 1\}$$



e set $X = C_1 \cup C_2$ is connected, since $(0,0) \in X$ so no matter how you to separate it into two non-empty separate sets A, B, either A $\overline{B}$ must have $(0,0)$, in which case $A \cap \overline{B} \neq \Phi$; OR $\overline{A}$ OR B must have)), in which case $\overline{A} \cap B \neq \Phi$; so there exists no such A, B and X is connected. that for this X, we have that $X^\circ = (C_1 \cup C_2)^\circ = C_1^\circ \cup C_2^\circ$. But and C_2° are two open disjoint sets; therefore by (19)(b) these are rate so that X° is not connected.

closures of connected sets are always connected.

: Let X be a connected set. We want to prove that $\overline{X}$ is connected

Suppose that $\overline{X}$ is not connected. Then, there exists separated, non-empty sets A, B, such that $\overline{X} = A \cup B$.

claim: $X = (A \cap X) \cup (B \cap X)$.

Pf: We can prove the equivalent statement: $X = X \cap (A \cup B)$.

($\supseteq$) let $x \in X \cap (A \cup B) \Rightarrow x \in X$, by definition of intersection

($\subseteq$) let $x \in X$. We want to prove $x \overset{?}{\in} (A \cup B)$. By definition of $A \cup B$:

$x \in X \Rightarrow x \in X \cup X' \Rightarrow x \in \overline{X} \Rightarrow x \in A \cup B$. ▫ (end of claim)

claim: $A \cap X = \Phi$ or $B \cap X = \Phi$. (Since C is connected)

Pf: Suppose $A \cap X \neq \Phi$ and $B \cap X \neq \Phi$ then these are separated since: $(A \cap X) \cup (\overline{B \cap X}) \subset (A \cap X) \cup (\overline{B} \cap \overline{X}) = (A \cup \overline{B}) \cap (\overline{B} \cup X) \cap (A \cup \overline{X}) \cap (X \cup \overline{X})$; but A and B are separated, thus $= \Phi \cap (\overline{B} \cup X) \cap (A \cup \overline{X}) \cap (X \cup \overline{X}) = \Phi$

$\Rightarrow (A \cap X) \cup (\overline{B \cap X}) \subset \Phi \Rightarrow (A \cap X) \cup (\overline{B \cap X}) = \Phi$. Likewise;

$(\overline{A \cap X}) \cup (B \cap X) \subset (\overline{A} \cap \overline{X}) \cup (B \cap X) = (\overline{A} \cup B) \cap (\overline{A} \cup X) \cap (B \cup \overline{X}) \cap (X \cup \overline{X})$; but A and B are separated, thus $= \Phi \cap (\overline{B} \cup X) \cap (A \cup \overline{X}) \cap (X \cup \overline{X}) = \Phi$

$\Rightarrow (\overline{A \cap X}) \cup (B \cap X) \subset \Phi \Rightarrow (\overline{A \cap X}) \cup (B \cap X) = \Phi$. ▫

Hence, $X = (A \cap X) \cup (B \cap X)$ and $A \cap X = \Phi$ or $B \cap X = \Phi$. Finally, we want to show that $A = \Phi$. If that is the case, then $\overline{X}$ would be a connected set.

Suppose $A \cap X = \Phi$. Recall that $X = (A \cap X) \cup (B \cap X)$. In this case $X = \Phi \cup (B \cap X) \Rightarrow X = B \cap X \Rightarrow X \subset B \Rightarrow \overline{X} \subset \overline{B}$. But recall that A and B are separated and thus disjoint, so $A = A \cap \overline{X} = A \cap (A \cup B) = (A \cap A) \cup (A \cap B) = A \cup \Phi = A$. And thus:

$A = A \cap \overline{X} \subset A \cap \overline{B} = \Phi \Rightarrow A \subset \Phi \Rightarrow A = \Phi$. The argument is symmetric if we choose $B \cap X = \Phi$.

Therefore, the set $\overline{X}$ is connected.

So, given a connected set X, its closure $\overline{X}$ is also connected.

-4) Let X be a metric space in which every infinite subset has a limit point. Prove that X is separable.

ote: As explained in exercise (22), a metric space is called separable if it ontains a countable dense subset. therefore, what we want to do is: iven a metric space in which every infinite subset has a limit point, rove that X contains a countable dense subset.

f: Let X be a metric space in which every infinite subset has a limit point ix $\delta > 0$. Pick $x_1 \in X$. If possible, pick $x_2 \in X$ s.t. $d(x_1, x_2) \geq \delta > 0$. Hence, $\neq x_2$. Having chosen $x_1, \dots, x_j \in X$, choose $x_{j+1} \in X$, if possible, so that $(x_i, x_{j+1}) \geq \delta$ for $i = 1, \dots, j$. Note that $x_i \neq x_j$ for all $i \neq j$.

laim: the process described before must stop after a finite # of steps.

f (of claim): Suppose that the process goes on forever.
t $S = \{x_1, x_2, \dots, x_n, \dots\}$; with x_k pick as described before for all k.
en S is an infinite set and $S \subseteq X$. By hypothesis S has a limit point
t that is not the case: Let x be a limit point of S. Consider two cases:
(i) $x \in S$. then $N_{\delta/2}(x) \setminus \{x\} \cap S = \emptyset$; since every point of S other than itself is at distance $\geq \delta$ from x. Hence, x is not a limit point.
(ii) $x \notin S$. then, If $N_{\delta/2}(x) \setminus \{x\} \cap S \neq \emptyset$; it only contains one point of S, $\Rightarrow$ $N_{\delta/2}(x) \setminus \{x\} \cap S \Rightarrow y \in N_{\delta/2}(x)$, $y \neq x$, $y \in S \Rightarrow d(y, x_k) \geq \delta$ $\forall x_k \in S$.
$\{x, y\} = N_{\delta/2}(x) \setminus \{x\} \cap S$. But this contradicts theorem 2.20 which states at every neighborhood of a limit point x contains infinitely many points of S.
refore, S does not have a limit point, which means that $|S| < \infty$
(End of Pf of claim)

refore, $S = \{x_1, x_2, \dots, x_n\}$; with x picked as explained. But the choice of x_i end δ. A better notation would be $S_\delta = \{x_1, x_2, \dots, x_n\}$.

im: Let $B = \bigcup_{n \in \mathbb{N}} S_{\delta_n}$; where $\delta_n = \frac{1}{n}$. then B is a countable, dense set in X.

i) B is countable since each S_{δ_n} is a finite set and the countable union of finite sets is countable. (follows from theorem 2.12).

i) B is dense in X. Let $x \in X$. If $x \in B$, we are done. otherwise, if B then x is a limit point of B. Let $r > 0$. Consider $N_r(x) \setminus \{x\} \cap B$. Choose $< r$ (which we can do by archimedian principle). then, there exists B s.t $d(y, x) < \delta_n < r \Rightarrow y \in N_r(x) \setminus \{x\} \cap B \Rightarrow x$ is a limit point of B.

(ii) $\Rightarrow$ B is a countable dense subset of $X \Rightarrow X$ is separable □

(26) Let X be a metric space in which every infinite subset has a l.p.
Prove that X is compact.

Pf: By exercises 23 and 24, X has a countable base. It follows that
every open cover of X has a countable subcover $\{G_n\}$, $n = 1, 2, 3, \ldots$
Suppose, for a contradiction, that X is not compact. then, no finite
subcollection of $\{G_n\}$ covers X.

Claim: Let $F_n = (G_1 \cup \cdots \cup G_n)^c$. then (i) $F_n \neq \Phi$ for $n \in \mathbb{N}$ and (ii) $\bigcap F_n = \Phi$

(i) Suppose $F_n = \Phi$. then $X = F_n^c = ((G_1 \cup \cdots \cup G_n)^c)^c = G_1 \cup \cdots \cup G_n$; so
F_n would be a finite subcover a contradiction.

(ii) this follows from the fact that $\{G_n\}$ is an open cover of X.
So every element $x \in X$ belongs to some G_{n_i} where G_{n_i} is an open set
from the cover. But F_n^c is everything that is not in a finite subcover
of $\{G_n\}$. Hence, eventually $n \to \infty$ all $x \in \bigcup_{n \in \mathbb{N}} G_n$ which means that
none will be in F_n for all n. thus, $\bigcap F_n = \Phi$.

(End of claim)

Let $E = \{f_1, f_2, \ldots, f_n, \ldots\}$, where $f_i \in F_n$; so ~~E~~ is a set which
contains a point from each F_n.

claim: E is infinite.

Pf: By previous claim we know that (i) $F_n \neq \Phi$, $n \in \mathbb{N}$ and (ii) $\bigcap F_n = \Phi$
If E was to be finite, then we would have to have an element f
such that $f \in F_n$ $\forall n$. But $\bigcap F_n = \Phi$; so no such f exists. moreover
we can always pick a distinct f_i because $F_n \neq \Phi$ for all n.

(end of claim)

Now, $E \subseteq X$ and E is infinite. By assumption E has a limit point.
Let $x \in X$ be a limit point of E. then, $\forall r > 0$: $N_r(x) \setminus \{x\} \cap E \neq \Phi$.
Let $y \in N_r(x) \setminus \{x\} \cap E$. then $y \in N_r(x)$, $y \neq x$, $y \in E$. Moreover, $N_r(x)$ contains
infinitely many points of E (infinitely many f_i's). But F_n is closed,
(since G_n is open), so $x \in F_n$, $\forall n$. therefore $x \in \bigcap F_n = \Phi$, a clear
contradiction. therefore, X is compact.